# Note for Nikiforov's two conjectures on the energy of trees*

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**Abstract**

The energy $E$ of a graph is defined to be the sum of the absolute values of its eigenvalues. Nikiforov in *"V. Nikiforov, The energy of $C_4$-free graphs of bounded degree, Lin. Algebra Appl. 428(2008), 2569–2573"* proposed two conjectures concerning the energy of trees with maximum degree $\Delta \leq 3$. In this short note, we show that both conjectures are true.

**Key words:** energy of a graph, conjecture, tree

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Let $G$ be a graph on $n$ vertices and $\lambda_1,\ \lambda_2, \cdots, \lambda_n$ be the eigenvalues of its adjacency matrix. The value $E(G) = |\lambda_1| + \cdots + |\lambda_n|$ is defined as the energy of $G$, which has been studied intensively, see [1, 3] for a survey.

In [5], Nikiforov proposed two conjectures on the energy of trees. In order to state and prove them, we need the following notations and terminology.

The *complete $d$-ary tree* of height $h-1$ is denoted by $C_h$, which is built up inductively as follows: $C_1$ is a single vertex and $C_h$ has $d$ branches $C_{h-1}, \cdots, C_{h-1}$. See Figure 1 for examples. It is convenient to set $C_0$ as the empty graph.

Let $\mathcal{T}_{n,d}$ be the set of all trees with $n$ vertices and maximum degree $d+1$. We define a special tree $T^*_{n,d}$ as follows (see also [4]):

**Definition 1** *$T^*_{n,d}$ is the tree with $n$ vertices that can be decomposed as in Figure 2*

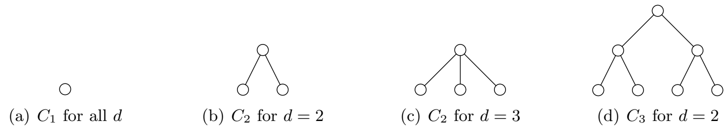


Figure 1  Some small complete $d$-ary trees.

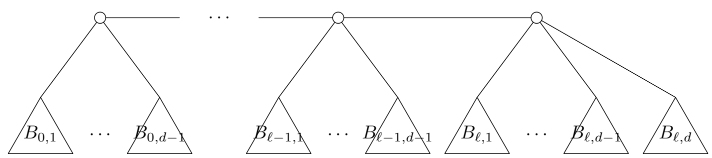


Figure 2  Tree $T^*_{n,d}$.

*with* $B_{k,1}, \cdots, B_{k,d-1} \in \{C_k, C_{k+2}\}$ *for* $0 \leq k < l$ *and either* $B_{l,1} = \cdots = B_{l,d} = C_{l-1}$ *or* $B_{l,1} = \cdots = B_{l,d} = C_l$ *or* $B_{l,1}, \cdots, B_{l,d} \in \{C_l, C_{l+1}, C_{l+2}\}$*, where at least two of* $B_{l,1}, \cdots, B_{l,d}$ *equal* $C_{l+1}$*. This representation is unique, and one has the "digital expansion"*

$$(d-1)n + 1 = \sum_{k=0}^{l} a_k d^k, \tag{1}$$

*where* $a_k = (d-1)(1 + (d+1)r_k)$ *and* $0 \leq r_k \leq d-1$ *is the number of* $B_{k,i}$ *that are isomorphic to* $C_{k+2}$ *for* $k < l$*, and*

- $a_l = 1$ *if* $B_{l,1} = \cdots = B_{l,d} = C_{l-1}$,
- $a_l = d$ *if* $B_{l,1} = \cdots = B_{l,d} = C_l$,
- *or otherwise* $a_l = d + (d-1)q_l + (d^2-1)r_l$*, where* $q_l \geq 2$ *is the number of* $B_{l,i}$

*that are isomorphic to* $C_{l+1}$ *and* $r_l$ *is the number of* $B_{l,i}$ *that are isomorphic to* $C_{l+2}$.

Let $\mathcal{B}_n$ denote the tree constructed by taking three disjoint copies of the complete 2-ary tree of height $h-1$, i.e., $C_n$, and joining an additional vertex to their roots (i.e., vertices of height zero). In the end of [5], Nikiforov formulated two conjectures as follows:

**Conjecture 2** *The limit*

$$c = \lim_{n\to\infty} \frac{E(\mathcal{B}_n)}{3 \cdot 2^{n+1} - 2}$$

*exists and* $c > 1$.

**Conjecture 3** *Let* $\epsilon > 0$*. If* $T$ *is a sufficiently large tree with* $\Delta(T) \leq 3$*, then* $E(T) \geq (c - \epsilon)|T|$.

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Nikiforov mentioned that empirical data given in [2] seem to corroborate these conjectures, but apparently new techniques are necessary to prove or disprove them. We will give confirmative proofs for both Conjecture 2 and Conjecture 3.

We first state two known lemmas from [4], which will be needed in the sequel.

**Lemma 4** *[4]. Let $n$ and $d$ be positive integers. Then $T^*_{n,d}$ is the unique (up to isomorphism) tree in $\mathcal{T}_{n,d}$ that minimizes the energy.*

**Lemma 5** *[4]. The energy of $T^*_{n,d}$ is asymptotically*

$$E(T^*_{n,d}) = \alpha_d \cdot n + O(\ln n),$$

*where*

$$\alpha_d = 2\sqrt{d}(d-1)^2 \left( \sum_{\substack{j \geq 1 \\ j \equiv 0 \pmod 2}} d^{-j} \left( \cot \frac{\pi}{2j} - 1 \right) + \sum_{\substack{j \geq 1 \\ j \equiv 1 \pmod 2}} d^{-j} \left( \csc \frac{\pi}{2j} - 1 \right) \right) \tag{2}$$

*is a constant that only depends on $d$.*

| $d$ | $\alpha_d$ |
|---|---|
| 2 | 1.102947505597 |
| 3 | 0.970541979946 |
| 4 | 0.874794345784 |
| 5 | 0.802215758706 |
| 6 | 0.744941364903 |
| 7 | 0.698315075830 |
| 8 | 0.659425329682 |
| 9 | 0.626356806404 |
| 10 | 0.597794680849 |
| 20 | 0.434553264777 |
| 50 | 0.279574397741 |
| 100 | 0.198836515295 |

Table 1 Some numerical values for the constant $\alpha_d$.

With the above two lemmas, the two conjectures can be proved very easily as follows.

**Theorem 6** *The limit*

$$c = \lim_{n \to \infty} \frac{E(\mathcal{B}_n)}{3 \cdot 2^{n+1} - 2}$$

*exists and $c > 1$.*

*Proof.* We just need to notice that $\mathcal{B}_n$ is exactly the tree $T^*_{3\cdot 2^{n+1}-2,2}$ with $l = n,\ B_{k,1} = C_k$ for $0 \leq k < l,\ B_{l,1} = B_{l,2} = C_n$. Therefore, by Lemma 5 and Table 1 we have

$$\lim_{n\to\infty}\left(\frac{E(\mathcal{B}_n)}{3\cdot 2^{n+1}-2}\right) = \lim_{n\to\infty}\left(\alpha_2 + \frac{O(\ln(3\cdot 2^{n+1}-2))}{3\cdot 2^{n+1}-2}\right) = \alpha_2 > 1.$$

∎

In fact, from Lemmas 4 and 5 we have that for any $T \in \mathcal{T}_{n,d}$,

$$E(T) \geq E(T^*_{n,d}) = \alpha_d \cdot n + O(\ln n).$$

Therefore, we obtain

**Theorem 7** *Let $\epsilon > 0$. If $T$ is a sufficiently large tree with $\Delta(T) = d+1$, then $E(T) \geq (\alpha_d - \epsilon)|T|$, where $\alpha_d$ is given in Equ.(2).*

Letting $d = 2$, we get

**Corollary 8** *Let $\epsilon > 0$. If $T$ is a sufficiently large tree with $\Delta(T) = 3$, then $E(T) \geq (\alpha_2 - \epsilon)|T|$, where $\alpha_2$ is given in Equ.(2).*

Recall that a hypoenergetic graph of order $n$ is such that $E(G) < n$, whereas it is strongly hypoenergetic if $E(G) < n-1$. We have the following easy remarks:

**Remark 1:** From Lemma 5 and Table 1, one can see that there is neither strongly hypoenergetic tree nor hypoenergetic tree of order $n$ and maximum degree $\Delta$ for $\Delta \leq 3$ and any suitable large $n$.

**Remark 2:** From Lemma 5 and Table 1, one can also see that there are both hypoenergetic trees and strongly hypoenergetic trees of order $n$ and maximum degree $\Delta$ for $\Delta \geq 4$ and any suitable large $n$.

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